OF THE ... SOCIETY

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PROGRESS REPORT.

1. The following gentlemen have been elected members of the Society—

(1) *Mr. Sadashiva B. Belekar, M.A.*—Assistant Teacher, Nasik High School, Nasik, (at concessional rate);

(2) *Mr. S. Balakrishna Aiyar, B.A., L.T.*—Assistant Lecturer, Kumbhakonam College, 17, Bhaktapuri Street, Kumbhakonam;

(3) *Mr. T. V. L. Narasimhachari, M.A., L.T.*—Lecturer in Mathematics and Sanskrit, Noble College, Masulipatam;

(4) *Mr. Y. S. Narayana, B.A.*—Mathematics Assistant, Noble College, Masulipatam; (at concessional rate);

(5) *Mr. E. R. Seshu Iyer, B.A.*—M.A. Student, Presidency College, 23, Sundaramurthi Vinayaka Koil Street, Triplicane, Madras, (at concessional rate).

2. The proposal for the formation of the Committee for 1913, has been voted upon favourably by the General Body, and the Committee for 1913 will therefore consist of the following—

1. Prof. R. N. Apte, M.A., L.L.B., F.R.A.S. ... *President.*
2. The Rev. C. Pollard, M.A. ... ... *Treasurer.*
3. Principal R. P. Paranjpye, M.A., B. Sc. ... *Librarian.*
4. Mr. Balak Ram, M.A., I.C.S.
5. Mr. K. Balaram Iyer, B.A.
6. Prof. Gopal Singh Chowla, M.A.
7. Mr. D. D. Kapadia, M.A., B.Sc.

8. Mr. M. P. Khareghat, I.C.S. (retired.)
9. Prof. M. T. Naraniengar, M.A.
10. Mr. S. Narayana Aiyar, M.A.
11. Mr. R. Swaminatha Iyer, B.A.
12. Prof. A. C. L. Wilkinson, M.A., F.R.A.S.

3. The following book has been presented to the Library—

"*Ganita-Sara-Sangraha* of Mahāvirachārya," with English Translation and Notes, by *Rao Bahadur* Professor M. Rangāchārya, M.A., Government Press, Madras, 1912.

POONA,
30th November, 1912.

D. D. KAPADIA,
Hony. Joint Secretary.

The Theory of Moving Axes.

By A. C. L. Wilkinson, M.A., F.R.A.S.

(*Continued from page 142.*)

§ 59. *Surfaces of which one set of lines of curvature are plane curves lying in parallel planes.*

Take the lines of curvature as parameters, and v = constant, the plane lines of curvature.

Since the torsion is zero, we have from § 44,

$$\tan \chi = \frac{1}{D}\sqrt{\frac{E}{G}}\frac{\partial\sqrt{E}}{\partial v} = \text{a function of } v \text{ only.}$$

Also, the binormal (0, $\cos\chi$, $\sin\chi$) must be fixed in space for variations of v, whence moving the trihedral along u = constant, for which

$$\partial_1 = \frac{D''}{\sqrt{G}}dv,\ \partial_2 = 0,\ \partial_3 = \frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}dv,$$

we have

$$-\cos\chi\partial_3 + \sin\chi\partial_2 = 0,$$
$$-\sin\chi d\chi - \sin\chi\partial_1 = 0,$$
$$\cos\chi d\chi + \cos\chi\partial_1 = 0;$$

whence

$$d\chi + \frac{D''}{\sqrt{G}}dv = 0 \text{ and } \cos\chi\partial_3 = 0.$$

But χ is a function of v only, therefore $\frac{D''}{\sqrt{G}}$ is a function of v only.

Case (i): Consider $\cos\chi = 0$, $\chi = 90$, $\tan\chi = \infty$, $D = 0$; we have also $d\chi = 0$, whence $D'' = 0$.

But if $D = D' = D'' = 0$, the curvatures at every point of the surface are both zero and the surface is a plane.

Case (ii): $\partial_3 = 0$, whence G is a function of v only.

By a change of variable v, we can take $G = 1$;

we have therefore $\frac{\sqrt{E}}{D}\frac{\partial\sqrt{E}}{\partial v}$ = a function of v only,

$$D'' = \text{a function of } v \text{ only.}$$

Also, from Gauss' equation $\frac{DD''}{\sqrt{E}} = -\frac{\partial^2\sqrt{E}}{\partial v^2}$, we derive

$$\frac{\partial^2\sqrt{E}}{\partial v^2} \div \frac{\partial\sqrt{E}}{\partial v} = \text{a function of } v \text{ only};$$

thus $\sqrt{E} = U_1V + U_2$, where U_1, U_2 are functions of u only and V is a function of v only.

The surface is given by $ds^2=(U_1V+U_2)^2du^2+dv^2,$
$$=(U-V)^2du^2+dv^2,$$
by a change of parameter u, where U, V are functions of u, v respectively.

We may determine the coordinates of any point on the surface in the form given by Darboux, as follows:

A transformation of the fixed coordinate axes does not alter E,F,G,D,D′,D″; taking the axis of z perpendicular to the parallel planes, and taking $z=v$, as their equation, the surface must be

$$ds^2=(U_1V+U_2)^2du^2+[\{\phi'(v)\}^2+1]dv^2;$$

whence
$$\left(\frac{\partial x}{\partial u}\right)^2+\left(\frac{\partial y}{\partial u}\right)^2=(U_1V+U_2)^2,$$
$$\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}+\frac{\partial y}{\partial u}\frac{\partial y}{\partial v}=0,$$
$$\left(\frac{\partial x}{\partial v}\right)^2+\left(\frac{\partial y}{\partial v}\right)^2=\{\phi'(v)\}^2.$$

Also, $D'=0$ gives
$$\frac{\partial^2 x}{\partial u\partial v}\bigg/\frac{\partial x}{\partial u}=\frac{\partial^2 y}{\partial u\partial v}\bigg/\frac{\partial y}{\partial u},$$
whence $\partial x/\partial u=f(u)\partial y/\partial u=\tan u\ \partial y/\partial u$, by change of parameter u. Thus we have
$$\partial x/\partial u=(U_1V+U_2)\sin u,\ \partial x/\partial v=\phi'(v)\cos u$$
$$\partial y/\partial u=(U_1V+U_2)\cos u,\ \partial y/\partial v=-\phi'(v)\sin u$$

Thus
$$x=\phi(v)\cos u+\int U_2\sin u\,du;$$
$$y=-\phi(v)\sin u+\int U_2\cos u\,du,$$
$$z=v.$$

Writing $[f(u)+f''(u)]$ for U_2 we have, finally, for the coordinates of any point on the surface
$$x=\phi(v)\cos u-f(u)\cos u+f'(u)\sin u,$$
$$y=-\phi(v)\sin u+f(u)\sin, u+f'(u)\cos u,$$
$$z=v.$$

A simple and elegant method of obtaining these equations is given in Goursat: *Cours D'Analyse*.

We may note some other properties:

The surface is of the form Edu^2+dv^2. Therefore the lines $u=$constant, are geodesics and being lines of curvature, are co-planar.

We find $E=[U_2-\phi(v)]^2$, $G=[\phi'(v)]^2$, $D=\dfrac{U_2-\phi(v)}{\phi'(v)}$, $D''=\dfrac{\phi''(v)}{\phi'(v)}$

The radius of curvature of $v=$constant, is given by
$$\frac{1}{\rho^2}=\frac{1}{[U_2-\phi(v)]^2}\left(1+\frac{1}{[\phi'(v)^2]}\right).$$

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whence $$\rho = \frac{[U_2-\phi(v)]\phi'(v)}{\sqrt{[1+\phi'(v)]^2}}$$

The radius of curvature of n=constant, is $\rho = D''/G$ = a function of v only; and since u=constant, are geodesics, any two arcs intercepted by two orthogonal trajectories are equal. Hence, the curves u=constant, are all repetitions of the same curve.

Lastly, the planes of u=constant envelop a cylinder:

For, the plane u=constant, is $x=o$, and the consecutive plane is
$$x-\sqrt{E}\,du+y\partial_3-z\partial_2=0,$$
where ∂_2, ∂_3 are taken for v=constant.

The direction cosines of the line of intersection of these planes are $(o, \partial_2, \partial_3)$, or $(o, \cos\chi_1 \sin\chi)$, where
$$\tan\chi = \frac{\partial_3}{\partial_2} = \frac{1}{D}\sqrt{\frac{E}{G}}\frac{\partial\sqrt{E}}{\partial v},$$
and this is parallel to the fixed direction of the binormals to v=constant.

Thus the planes u=constant, envelop a cylinder; and since they are all repetitions of the same curve, the surface can be generated by a curve in a plane which rolls on a cylinder.

§ 60. *The transformation by inversion.*

Taking the centre of inversion as fixed origin, we write
$$x_1=\frac{ax}{R},\ y_1=\frac{ay}{R},\ z_1=\frac{az}{R},$$
where R is $x^2+y^2+z^2$, and a is the square of the radius of inversion.

The surface to be inverted is given by
$$ds^2=Edu^2+2Fdudv+Gdv^2,\ D, D', D''$$
and we shall denote the surface obtained by inversion by
$$ds_1^2=E_1du^2+2F_1dudv+G_1\,dv^2,\ D_1, D_1', D_1''.$$

Now $$E_1=\sum\left(\frac{\partial x_1}{\partial u}\right)^2=\sum\left(\frac{a}{R}\frac{\partial x}{\partial u}-\frac{ax}{R^2}\frac{\partial R}{\partial u}\right)^2=\frac{a^2}{R^2}E.$$

So $$F_1=\frac{a^2}{R^2}F,\ G_1=\frac{a^2}{R^2}G_1 \text{ and } H_1=\frac{a^2}{R^2}H.$$

For D_1 we have the expressions
$$\frac{1}{H_1}\left|\begin{array}{ll} \frac{a}{R}\frac{\partial^2 x}{\partial u^2}-\frac{2a}{R^2}\frac{\partial x}{\partial u}\frac{\partial R}{\partial u}+\frac{2ax}{R^3}\left(\frac{\partial R}{\partial u}\right)^2-\frac{ax}{R^2}\frac{\partial^2 R}{\partial u^2}, \ldots, & \ldots \\ \frac{a}{R}\frac{\partial x}{\partial u}-\frac{ax}{R^2}\frac{\partial R}{\partial u}, & \ldots\ \ldots \\ \frac{a}{R}\frac{\partial x}{\partial v}-\frac{ax}{R^2}\frac{\partial R}{\partial v}, & \ldots\ \ldots \end{array}\right|$$

$$=\frac{1}{H_1}\begin{vmatrix} \frac{a}{R}\frac{\partial^2 x}{\partial u^2}-\frac{ax}{R^2}\frac{\partial^2 R}{\partial u^2}, & \ldots, & \ldots \\ \frac{a}{R}\frac{\partial x}{\partial u}-\frac{ax}{R^2}\frac{\partial R}{\partial u}, & \ldots, & \ldots \\ \frac{a}{R}\frac{\partial x}{\partial v}-\frac{ax}{R^2}\frac{\partial R}{\partial v}, & \ldots, & \ldots \end{vmatrix} = \frac{a}{R}D-\sum\frac{1}{H}\begin{vmatrix} \frac{\partial^2 R}{\partial u^2}, \frac{\partial^2 y}{\partial u^2}, \frac{\partial^2 z}{\partial u^2} \\ \frac{\partial R}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \\ \frac{\partial R}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \end{vmatrix}\frac{ax}{R^2}$$

$$=\frac{a}{R}D-\sum\frac{1}{H}\begin{vmatrix} 2x\frac{\partial^2 x}{\partial u^2}+2y\frac{\partial^2 y}{\partial u^2}+2z\frac{\partial^2 z}{\partial u^2}+2E, & \frac{\partial^2 y}{\partial u^2}, \frac{\partial^2 z}{\partial u^2} \\ 2x\frac{\partial x}{\partial u}+2y\frac{\partial y}{\partial u}+2z\frac{\partial z}{\partial u}, & \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \\ 2x\frac{\partial x}{\partial v}+2y\frac{\partial y}{\partial v}+2z\frac{\partial z}{\partial v}, & \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \end{vmatrix}\frac{ax}{R^2}$$

$$=\frac{a}{R}D-2\frac{a}{R}D-\frac{2a}{R^2}\frac{E}{H}\sum x\frac{\partial(y,z)}{\partial(u,v)}=-\frac{a}{R}D-\frac{2a}{R^2}\frac{E}{H}\sum x\frac{\partial(y,z)}{\partial(u,v)}$$

Thus we have

$$ds_1^2=\frac{a^2}{R^2}(Edu^2+2Fdu\,dv+Gdv^2)$$

$$D_1=-\frac{a}{R}\left\{D+\frac{2E}{RH}\sum x\frac{\partial(y,z)}{\partial(u,v)}\right\}$$

$$D_1'=-\frac{a}{R}\left\{D'+\frac{2F}{RH}\sum x\frac{\partial(y,z)}{\partial(u,v)}\right\}$$

$$D_1''=-\frac{a}{R}\left\{D''+\frac{2G}{RH}\sum x\frac{\partial(y,z)}{\partial(u,v)}\right\}$$

If $F=0$, $D'=0$; then also $F_1=0$, $D'_1=0$, whence we have Darboux's theorem :

When a surface is inverted from any point its lines of curvature inver into the lines of curvature of the inverse surface.

We deduce also

$$F_1D_1-E_1D_1'=-\frac{a^3}{R^3}(FD-ED')$$

$$F_1D_1''-G_1D_1'=-\frac{a^3}{R^3}(FD''-GD').$$

Thus $$t_1=\frac{E_1D_1'-F_1D_1}{H_1E_1}=-\frac{Rt}{a},$$

and $$t_1'=-\frac{Rt'}{a}.$$

Also $n_1-n_1'=\frac{D_1}{E_1}-\frac{D_1''}{G_1}=-\frac{R}{a}\left(\frac{D}{E}-\frac{D''}{G}\right)=-\frac{R}{a}(n-n')$; and in particular if ρ_1, ρ_2, ρ_1', ρ_2' are the principal radii of curvature of a surface and its inverse at corresponding points,

$$\frac{1}{\rho_1'}-\frac{1}{\rho_2'}=-\frac{R}{a}\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right).$$

As there is an ambiguity of sign here owing to $n=\frac{\cos\psi}{\rho}$, it is better perhaps to deduce the result from the eqnations giving ρ_1, ρ_2, ρ_1', ρ_2', which are

$$(DD''-D'^2)\,\rho^2-(ED''+GD-2\,FD')\rho+EG-F^2=0,$$

and the equation got by writing $E_1 \ldots D_1 \ldots$ for $E \ldots D \ldots$

A simple calculation will show that we have

$$\frac{1}{\rho_1'}\sim\frac{1}{\rho_2'}=\frac{R}{a}\left(\frac{1}{\rho_1}\sim\frac{1}{\partial_3}\right)=\frac{a}{R_1}\left(\frac{1}{\rho_1}\sim\frac{1}{\rho_2}\right),\ \text{where } R_1^2=x_1^2+y_1^2+z_1^2.$$

Consider, for example, the circular cylinder $x^2+y^2=c^2$, inverted with respect to the origin and the sphere $(x^2+y^2+z^2)=c^2$.

We obtain for the inverse surface $(x^2+y^2+z)^2=c^2(x^2+y^2)$, the theorem

$$\frac{1}{\rho_1}\sim\frac{1}{\rho_2}=\frac{c}{x^2+y^2+z^2},$$

as may be easily verified geometrically or otherwise.

§. 61. *Developable surfaces.*

Take the generators as the lines $v=$constant, and let u be the distance to any point measured along a generator from the point in which the generator is met by one of the orthogonal trajectories.

Take $x=b_1, y=b_2, z=b_3$, as the equations of the fixed orthogonal trajectory, where the b's are functions of v. Let a_1, a_2, a_3 be the direction cosines of the generator through the point (b_1, b_2, b_3), the a's are also functions of v.

The generator is therefore

$$x=a_1u+b_1\quad y=a_2u+b_2,\ z=a_3u+b_3,$$

where $\qquad a_1^2+a_2^2+a_3^2=1$ and $a_1a_1'+a_2a_2'+a_3a_3'=0$ ... ... (1)

and since $u=0$, is the orthogonal trajectory of the generators, we have

$$a_1b_1'+a_2b'_2+a_3b_3'=0, \ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

where accents refer to differentiation with respect to v.

Thus for the element of the surface

$$ds^2=du^2+\{(a'u+b_1')^2+(a_2'u+b_2')^2+(a_3'u+b_3')^2\}\,dv^2.$$

So far this applies to all ruled surfaces. For a developable, if u is the distance to the point where the generator meets the edge of regression, u will be a function of v, and the tangent to the edge of regression must coincide with the generator: we have therefore

$$\frac{a_1u'+a_1'u+b_1}{a_1}=\frac{a_2u'+a_2'u+b'}{a_2}=\frac{a_3u'+a_3'u+b_3'}{a_3}=u',$$

on multiplying by a_1, a_2, a_3 and adding and using conditions (1) and (2).

Whence $-u=b_1'/a_1'=b_2'/a_2'=b_3'/a_3'=k$, a function of v only, defines the edge of regression.

Thus a developable surface is given by

$$ds^2 = du^2 + (a_1'^2 + a_2'^2 + a_3'^2)(u+k)^2 dv^2;$$

we can replace v by $\int\sqrt{a_1'^2 + a_2'^2 + a_3'^2}\,dv$, and write the element of the developable in the standard form

$$ds^2 = du^2 + (u+k)^2 dv^2,$$

where k is a function of v only; and in addition to the equations (1) and (2) we have

$$a_1'^2 + a_2'^2 + a_3'^2 = 1, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

and $$b_1'/a_1' = b_2'/a_2' = b_3'/a_3' = k, \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

Thus $E=1$, $F=0$, $G=(u+k)^2$, $H=u+k$.

We proceed to calculate D, D', D''

$D=0.$

$$D' = \frac{1}{u+k}\begin{vmatrix} a_1', & a_2', & a_3' \\ a_1, & a_2, & a_3 \\ a_1'u+b_1', & a_2'u+b_2', & a_3'u+b_3' \end{vmatrix} = 0, \text{ since } b_1' = ka_1', \text{ etc.}$$

$$D'' = \frac{1}{u+k}\begin{vmatrix} a_1''u+b_1'', & a_2''u+b_2'', & a_3''u+b_3'' \\ a_1, & a_2, & a_3 \\ a_1'u+b_1', & a_2'u+b_2', & a_3'u+b_3' \end{vmatrix}$$

which reduces, since $b_1'' = ka_1'' + k'a_1'$, to

$$D'' = (u+k)\begin{vmatrix} a_1'', & a_2'', & a_3'' \\ a_1, & a_2, & a_3 \\ a_1' & a_2', & a_3' \end{vmatrix} = A(u+k),$$

where A is a function of v only. We find, by squaring A, that $A^2 = a_1''^2 + a_2''^2 + a_3''^2 - 1$.

Consider the edge of regression: we move the trihedral along the orthogonal trajectory $u=o$, whence

$$\xi = 0,\ \eta = k\,dv,\ \zeta = 0,$$
$$\partial_1 = A\,dv,\ \partial_2 = 0,\ \partial_3 = dv.$$

The tangent is $(1,0,0)$ and the consecutive tangent is $(1,\partial_3,0)$ whence for the contingent angle $d\varepsilon^2 = dv^2$.

Also, the element of arc of the edge of regression is dk.

Thus $\rho = \dfrac{dk}{dv}$ gives the radius of curvature at the point $(-k,o,o)$ on the edge of regression.

The binormal is $(0,0,1)$ and the consecutive binormal $(0,-\partial_1,1)$, whence

$$dT^2 = A^2 dv^2$$

and $$\sigma = \frac{A\,dv}{dk} = \frac{A}{\rho}.$$

The principal radius of curvature at any point of the developable which is not zero is given by

$$R=\frac{G}{D''}=\frac{u+k}{A}.$$

But $u+k$ is the distance of the point from ths edge of regression measured along the generator, whence we derive a theorem stated in Eisenhart, p. 149.

The only developable surface whose lines of curvature are plane are those whose edge of regression is a helix on a general cylinder.

For, the radius of torsion of the lines of curvature u=constant, is given by

$$\frac{1}{\sigma}=-\frac{1}{u+k}\frac{\partial X'}{\partial v},\text{ where }\tan X'=\frac{1}{A}.$$

Thus we must have X' constant and therefore A independent of v.

Thus for the edge of regression ρ/σ=constant, which is the fundamental property of the general helix.

The equation of the geodesics on a developable is

$$d\theta+\partial_s=0$$

or

$$\theta+v=\text{constant};$$

and since $\tan\theta=\frac{(u+k)dv}{du}$, we have for the differential equation of the geodesics,

$$\frac{du}{dv}=(u+k)\cot(\alpha-v),$$

where α is the constant angle $\theta+v$.

§. 62. *Ruled surfaces*:

If $x=a_1u+b_1$, $y=a_2u+b_2$, $z=a_3u+b_3$, define the generators of a ruled surface, we saw in the previous section that the element can be put in the form

$$ds^2=du^2+\{(u-\alpha)^2+\beta^2\}dv^2,$$

where

$$a_1^2+a_2^2+a_3^2=1=a_1'^2+a_2'^2+a_3'^2$$

$$a_1b_1'+a_2b_2'+a_3b_3'=0$$

$$\alpha=-(a_1'b_1'+a_2'b_2'^2+a_3'b_3')$$

$$\beta^2=b_1'^2+b_2'^2+b_3'^2-(a_1'b_1'+a_2'b_2'+a_3'b_3')^2$$

$$=\Sigma(a_1'b_2'-a_2'b_1')^2,\text{ by use of }a_1'^2+a_2'^2+a_3'^2=1.$$

We find $D=0$

$$D'=\frac{1}{\sqrt{G}}\begin{vmatrix} a_1' & a_2' & a_3' \\ a_1 & a_2 & a_3 \\ b_1' & b_2' & b_3' \end{vmatrix}$$

whence $$D'^2 = \frac{1}{(u-\alpha)^2+\beta^2} \begin{vmatrix} 1 & 0 & \Sigma a_1'b_1' \\ 0 & 1 & 0 \\ \Sigma a_1'b_1' & 0 & \Sigma b_1'^2 \end{vmatrix} = \frac{\beta^2}{(u-\alpha)^2+\beta^2}$$

Thus for the orthogonal trajectories of the generating lines to form the lines of curvature, we have $\beta=0$ and, from the expression for β given above, this gives $a_1'/b_1'=a_2'/b_2'=a_3'/b_3'$, the conditions for a developable.

We can take $D'=\beta/\sqrt{G}$, as it will appear immediately that the sign of D' simply determines the direction in which the generators twist round.

Hence for a point on $u=$constant, when the trihedral is displaced along the orthogonal trajectory to the generators, we have

$$\partial_1 = \frac{D''}{\sqrt{G}}dv,\ \partial_2 = -D'dv,\ \partial_3 = \frac{\partial\sqrt{G}}{\partial u},$$

or

$$\partial_1 = \frac{D''}{\sqrt{G}}dv,\ \partial_2 = -\frac{\beta}{\sqrt{G}}dv,\ \partial_3 = \frac{u-\alpha}{\sqrt{G}}dv,$$

where $$G=\sqrt{(u-\alpha)^2+\beta^2}$$

To determine the line of striction :

Taking moving axes displaced along the orthogonal trajectory of the generators, two consecutive generators are

$$\frac{x}{1}=\frac{y}{0}=\frac{z}{0},$$

and $$\frac{x}{1}=\frac{y-\sqrt{G}\,dv}{\partial_3}=\frac{z}{-\partial_2};$$

or $$\frac{x}{1}=\frac{y-\sqrt{\alpha^2+\beta^2}dv}{-\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}dv}=\frac{z}{\frac{\beta}{\sqrt{\alpha^2+\beta^2}}dv},$$ where we put $u=0$.

If now $\frac{x-k}{0}=\frac{y}{\cos\chi}=\frac{z}{\sin\chi}$ is the equation ot the shortest distance,

then, $$-\alpha\cos\chi+\beta\sin\chi=0,$$

and $$\begin{vmatrix} k, & -\sqrt{\alpha^2+\beta^2}\,dv & 0 \\ 0, & \beta & \alpha \\ 1, & -\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}dv, & \frac{\beta}{\sqrt{\alpha^2+\beta^2}}dv \end{vmatrix}=0.$$

$$\therefore\quad k(\alpha^2+\beta^2)=(\alpha^2+\beta^2)\alpha$$

or $$k=\alpha.$$

Further, the magnitude of the shortest distance is the projection of $\sqrt{\alpha^2+\beta^2}dv$ on the common perpendicular of the generators.

Therefore the shortest distance $=\sqrt{\alpha^2+\beta^2}\,dv\cos\chi=\beta dv$;

Also, the angle between the generators is $\sqrt{\partial_3^2+\partial_2^2}=dv$;

Thus β which is the ratio between the shortest distance between two consecutive generators to the angle between them is the *parameter of distribution.*

On the tangent planes at different points of a generator.

Take the orthogonal trajectory $u=o$, and axes there.

Any point on the generator is (u,o,o), a consecutive point on this generator is $(u+du,o,o)$. A third point to determine the tangent plane is the point on the consecutivee generator which is

$$(u+du,\ \sqrt{G}\ dv+u\partial_3, -u\partial_2)$$

or
$$\left(u+du, \sqrt{\alpha^2+\beta^2}\ dv-\frac{u\alpha}{\sqrt{\alpha^2+\beta^2}}dv,\ \frac{u\beta}{\sqrt{\alpha^2+\beta^2}}dv\right)$$

since $\sqrt{G_1}, \partial_2, \partial_3$ are to be taken for $u=o$.

The tangent plane reduces to

$$-y\ u\beta+z\left\{\beta^2-\alpha(u-\alpha)\right\}=0.$$

For the tangent plane at the point where the generator meets the line of striction for which $\alpha=u$, we have

$$y\alpha-z\beta=0.$$

Thus if θ is the angle between the tangent plane at the origin and the tangent plane at the point on the line of striction

$$\cos\theta=-\beta/\sqrt{\alpha^2+\beta^2} \text{ and } \tan\theta=-\alpha/\beta.$$

Thus we may write $\tan\theta=(u-\alpha)/\beta$. for the angle between the tangent planes at a point whose parameter is u and the corresponding point on the line of striction. This result is due to Chasles.

The normal at $(u,0,0)$ is

$$\frac{x-u}{0}=\frac{y}{-u\beta}=\frac{z}{\beta^2-\alpha(u-\alpha)}.$$

Thus the normals along the generator generate the hyperbolic paraboloid

$$xz\beta+y(\beta^2+\alpha^2-\alpha x)=0.$$

§. 63. The following theorem is due to Bonnet.

If a curve on a ruled surface has two of the following properties: (1) it is a line of striction, (2) it cuts the generators at a constant angle, (3) it is a geodesic line, it possesses the third property.

Consider a curve through the point $(u,0,0)$ which cuts the generators at an angle θ. Take the moving axes at the point and displace them along the orthogonal trajectory through the point.

The condition for a geodesic is $d\theta+\theta_3=0$,

or
$$d\theta+\frac{u-\alpha}{\sqrt{G}}dv=0$$

The condition for a line of striction is $u=\alpha$, whence Bonnet's theorem follows.

§. 64. *If the orthogonal trajectories of the generators are the asymptotic lines, the surface is a right helicoid.*

We require to calculate D″

$$D''=\frac{1}{\sqrt{G}}\begin{vmatrix} a_1''u+b_1'', & a_2''u+b_2'', & a_3''u+b_3'' \\ a_1, & a_2, & a_3 \\ a_1'u+b_1', & a_2'u+b_2', & a_3'u+b_3' \end{vmatrix}$$

Multiply this by $\begin{vmatrix} a_1' & a_2' & a_3' \\ a_1 & a_2 & a_3 \\ b_1' & b_2', & b_3' \end{vmatrix}=\pm\beta,$

and since the condition the coordinates be asymptotic lines is $D''=0$, we have

$$\begin{vmatrix} \Sigma b_1''a_1', & 0, & u-\alpha \\ -(u-\alpha), & 1, & 0 \\ \Sigma a_1''b_1'u+\Sigma b_1'b_1'' & 0, & -\alpha(u-\alpha)+\beta^2 \end{vmatrix}=0$$

where we have made use of the relations

$$\Sigma a_1a_1'=0,\quad \Sigma a_1a_1''=-1,\quad \Sigma a_1'a_1''=0,$$
$$\Sigma a_1b_1'=-\alpha,\qquad \Sigma a_1'b_1'=0.$$

If this vanishes for all values of u

$$\Sigma a_1''b_1'=0,\ \Sigma b_1''a_1'=0,\ \Sigma b_1'b_1''=0.$$

Thus α, β are both constant.

Replacing $u-\alpha$ by u, or referring the surface to the line of striction, we may write these conditions;

$$\Sigma a_1^2=1,\ \Sigma a_1b_1'=0,\ \Sigma a_1'b_1''=0,\ \Sigma b_1'b_1''=0,$$
$$\Sigma a_1'^2=1,\ \Sigma a_1'b_1'=0,\ \Sigma a_1''b_1'=0,\ \Sigma b_1'^2=\text{constant}.$$

By differentiation $\Sigma a_1b_1'=0$, $\Sigma a_1b_1''=0$, $\Sigma a_1b_1'''=0$, whence the line of striction has zero torsion and is therefore plane.

Take the plane $z=0$, of the fixed system of axes, to contain the line of striction, we have $b_3=0$, hence

$$a_1'b_1'+a_2'b_2'=0,\ a_1b_1'+a_2b_2'=0$$

Thus $a_1=ca_2$, where c is constant.

Therefore $cb_1'+b_2'=0$, and $cb_1+b_2=\text{constant}$.

The line of striction is therefore a straight line, which we can take for the axis of x. Thus $b_2=0$, and since $\Sigma b_1'^2=\text{constant}$, we have

$$b_1=kv,\ b_2=0,\ b_3=0,\ a_1=0,\ \text{where } k \text{ is constant.}$$

And $a_2^2+a_3^2=1=a_2'^2+a_3'^2$, give $a_2=\cos(v-\alpha)$, $a_3=\sin(v-\alpha)$, whence the generators are

$$x=k(v+\alpha),\ y=u\cos v,\ z=u\sin v\,;$$

and these are the equations of a right helicoid.

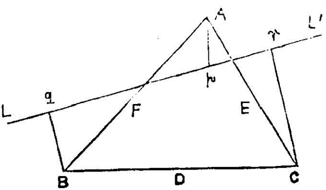
THE ORTHOPOLAR THEORY.

By V. Ramaswami Aiyar.

I. General Properties of the Orthopolar Circle.

1. Let ABC be a given triangle and D,E,F the middle points of its sides. Let LL′ be any straight line; and Ap, Bq, Cr perpendiculars drawn thereto from the vertices.

Then there is a circle, centre D, which passes through q, r; one with centre E, which passes through r, p; and one with centre F, which passes through p, q. Denote these circles by X, Y, Z. The centres of these circles are not collinear. There is therefore a single circle, Σ, which cuts these orthogonally.



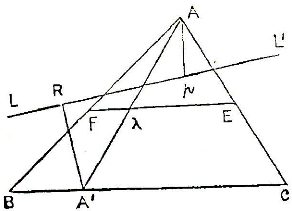
We shall call Σ the *orthopolar circle* of the line LL′ with respect to the triangle ABC.

2. Consider now the radical axes of the circles X, Y, Z taken in pairs. The circles Y and Z have their centres at E and F and pass through the point p. Hence if p' be the reflexion of p in EF, X and Y also pass through p', and pp' is their radical axis. Similarly if q',r' be the reflexions of q, r in FD, DE respectively, qq' and rr' are the radical axes of the pairs Z, X and X, Y. Hence, by coaxal circles, pp', qq' and rr' are concurrent at the centre of the orthopolar circle Σ.

This shows (1) that if Ap, Bq, Cr be perpendiculars on any line LL′, the perpendiculars from p, q, r on the sides BC, CA, AB are concurrent. The point of concurrence is *defined* as the orthopole of LL′ with respect to the triangle ABC. It follows (2) that the orthopole is the centre of the orthopolar circle. We further see (3) that if p', q', r' be the reflexions of p, q, r in the corresponding sides of the medial triangle the pairs p,p' ; q,q' ; r,r' ; are inverse points with respect to the orthopolar circle.

3. If the line LL′ under consideration becomes coincident with one of the sides of the triangle ABC it may be readily seen from the definitions that its orthopole is the orthocentre of ABC and its orthopolar circle is the polar circle of ABC.

4. *Theorem: If A′ be any point in BC and A′R be drawn perpendicular to LL′. the circle whose centre is the middle point of AA′ and which passes through R cuts the orthopolar circle of LL′ orthogonally.*



For Ap, A′R being perpendicular to LL′ the circle in question passes also through p; and its centre, λ, lies in EF. Hence it is coaxal with the circles Y and Z; and therefore, with Y and Z, it cuts the orthopolar circle, Σ, orthogonally.

5. Before proceeding further we shall state a few well-known properties of conics which are of use presently.

(1) Given two confocal (central) conics, the locus of the intersection of two rectangular tangents drawn one to each of them, is a circle concentric with the conics. It is called the *mutual orthoptic circle* of the two confocals.

(2) If a (central) conic be inscribed in a quadrilateral, its centre lies on the line passing through the middle points of the diagonals. (Newton's Theorem).

(3) The locus of the intersection of rectangular tangents, drawn one to each of two confocal parabolas, is the straight line lying midway between their directrices.

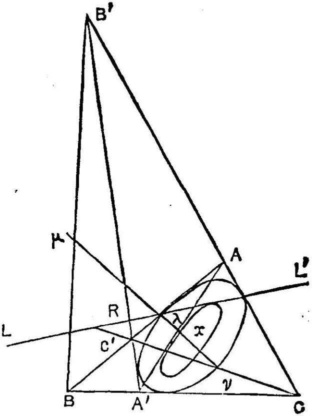
(4) If a parabola be inscribed in a quadrilateral, its axis is parallel to the line passing through the middle points of the diagonals.

All these theorems are easily proved geometrically. (3) and (4) also follow as limiting cases of (1) and (2). This is obvious as regards (4). As regards (3), it may be shown that the mutual orthoptic circle of two central confocals bisects the area of the *annulus* bounded by their director circles. Observing this, (3) is seen to be a limiting case of (1).

6. We can now enunciate a very general theorem:

If X be any conic inscribed in the triangle ABC and Y a confocal to it touching LL′, then, the mutual orthoptic circle of X and Y cuts the orthopolar circle of LL′ orthogonally.

Let a tangent be drawn to X so as to be perpendicular to LL′. Let it cut the sides BC, CA, AB in A′, B′, C′; and LL′ in R.



Then, by Art. 4, the circles whose centres are the middle points λ, μ, ν of AA′, BB′, CC′, and which pass through R, cut Σ orthogonally. Call these circles U, V, W. Their centres are collinear,—being the middle points of the diagonal of a quadrilateral. Hence they are coaxal. Let x be the centre of the conic X. By Newton's Theorem x lies in a line with λ, μ, ν. Hence the circle whose centre is x and which passes through R is coaxal with the circles U, V, W; and therefore, like them, it cuts Σ orthogonally. But R being an intersection of rectangular tangents drawn one to X and one to Y, the circle whose centre is x and which passes through R is the mutual orthoptic circle of X and Y. Therefore, the mutual orthoptic circle of X and Y cuts the orthopolar circle Σ orthogonally.

7. When the inscribed conic X becomes a parabola the theorem of the last article becomes the following:

If X be a parabola inscribed in a triangle ABC, and Y a confocal parabola touching LL′, then the line midway between their directrices passes through the orthopole of LL′.

A direct proof of this can be had by using properties (3) and (4) of Art. 5, and following the lines of the proof in the last article.

8. Let the line LL′ coincide with one of the sides of the triangle ABC. Remembering the observation in Art. 3, the theorems of articles 6 and 7 give us the following well known theorems :

(*a*) If any conic be inscribed in a triangle its director circle cuts the polar circle of the triangle orthogonally ; and

(*b*) If any parabola be inscribed in a triangle its directrix passes through the orthocentre.

9. Let LL′ be general in position. But let the inscribed conic X have one focus P on LL′. Then the confocal Y touching LL′ becomes the *thin ellipse joining P to the other focus of X*. And the mutual orthoptic circle of X and Y becomes the auxiliary circle of X ; that is, the pedal circle of P with respect to the triangle ABC. Hence we have the important theorem :

The pedal circle of every point on LL′ cuts the orthopolar circle of LL′ orthogonally.

The theorem that the pedal circles of points lying on a line cut a fixed circle orthogonally, due to LEMOYNE, and given by Mr. Gallatly to our readers (J. I. M. S., Vol. I, p. 228) is the origin of the present paper.

10. Let the line LL′ in the theorem of the last article cut the circumcircle of ABC in the points P,Q. Of the pedal circles of points on LL′ there are *three* which develop into straight lines. Two of these are obvious, namely, those corresponding to P and Q ; and we infer that the Simson lines of P and Q pass through the orthopole of LL′. The third point will also become obvious if we reflect that the pedal circles of two isogonally conjugate points are *identical*. Hence, if R be the isogonal conjugate of the point at infinity on LL′, we observe it lies on the circumcircle, and we infer that the Simson line of R passes through the orthopole of LL′. Hence we have the important theorems :

(1) *If a line LL′ cuts the circumcircle in P,Q the orthopole of LL′ is the point of intersection of the Simson lines of P and Q.*

(2) *If P, Q be any two points on the circumcircle of ABC, and R be the isogonal conjugate of the point at infinity on PQ, then the Simson lines of P, Q and R are concurrent.*

The properties in the present article can also be obtained, by means of the theorem of Art. 7, by taking the inscribed parabola X so as to have its focus successively at P,Q,R. We leave it to the reader to think this out completely.

(*To be continued.*)

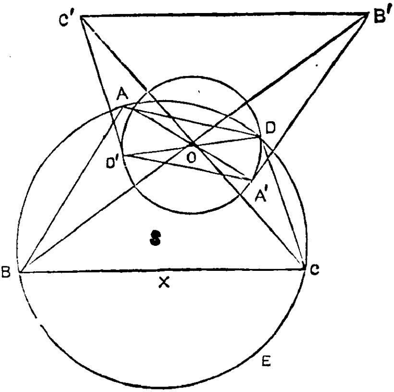
SHORT NOTES.

Note on a Cyclic Quadrilateral.

1. In a cyclic quadrilateral ABCD, let A′, B′, C′, D′ be the ortho-centres of the triangles BCD, CDA, DAB, ABC.

Then AD′ = 2S× = A′D, and since AD′ and A′D are also parallel, AD and A′D′ are equal and parallel. Similarly, AB and A′B′, BC and B′C′, CD and C′D′, are equal and parallel. Therefore AA′, BB′, CC′, DD′ bisect one another at the same point O.

Hence the quadrilateral A′B′C′D′ is equal to ABCD and similarly situated to it.



2. Since O is the middle point of AA′, BB′, CC′, DD′, the pedal line of each of the points A,B,C,D, with reference to the triangle formed by the other three passes through O.

3. The points A,B,C,D are clearly the ortho-centres of the triangles B′C′D′, C′D′A′, D′A′B′, A′B′C′. The point O has, therefore, the same relation to A′B′C′D′ that it has to ABCD. Hence the pedal line of each of the points A′,B′,C′,D′ with reference to the triangle formed by the other three passes through O.

4. It is readily seen that the Nine-point circles of the eight triangles ABC, A′B′C′........pass through O. As these circles are of the same dimensions, their *eight centres* must lie on an equal circle of which O is the centre. Call this the *eight-point* circle of ABCD.

5. If a fifth point E be taken on the circumference of the circle ABCD, the points corresponding to O in the quadrilaterals EBCD, ECDA, EDAB, EABC are respectively the middle points of EA′, EB′, EC′, ED′. These points form with O the angular points of a figure similar to ABECD and of half its linear dimensions. These five points lie, therefore, on a circle of the same dimensions as the circle of § 4. (*Vide*: Lachlan's 'Modern Geometry', *p.* 69.) Since this circle passes through O, the locus of its centre is the *eight-point* circle of ABCD.

17*th August*, 1912. M. KANNAN.

A generalized form of Clairaut's Equation.

1. The differential equation

$$y = mxp + ky^a p^b \quad \dots \quad \dots \quad \dots \quad (1)$$

is always soluble.

For, multiplying both sides of (1) by y^c, we have

$$y^{1+c} = mxy^c p + ky^{a+c} p^b.$$

Put $y^{1+c} = z$; then $(1+c)y^c p = q$, where $\frac{dz}{dx} = q$.

$$\therefore \ y^c p = \frac{q}{1+c}.$$

$$\therefore \ z = \frac{mx}{1+c} q + ky^{a+c} \left(\frac{q}{1+c}\right)^b + \frac{1}{y^{bc}}.$$

$$\therefore \ z = \frac{mx}{1+c} q + ky^{a+c-bc} \left(\frac{q}{1+c}\right)^b.$$

Select c so as to make the index of y vanish; thus $a+c=bc$.

$$\therefore \ c = \frac{a}{b-1}.$$

$$\therefore \ z = \frac{m\ (b-1)}{a+b-1}\, q\, x + k \left\{ q\, \frac{b-1}{a+b-1} \right\}^b \quad \dots \quad (2)$$

This comes under the standard form

$$z = x\, f\ (q) + \phi\ (q),$$

which becomes a linear differential equation of the first order in q and x after one differentiation with respect to x is performed.

Equation (2) will be in Clairaut's form if

$$m\ (b-1) = a + b - 1,$$

i.e., $$(m-1)\ (b-1) = a \quad \dots \quad \dots \quad (3)$$

If condition (3) is satisfied, the integral of the equation can be put in the form

$$y^m = Ax + k \{ A/m \}^b,$$

A being an arbitrary constant.

Examples:—

(i) As a particular case, consider the differential equation

$$p^3-4xyp+8y^2=0$$

This can easily be put in the form

$$y=\tfrac{1}{2}xp-\tfrac{1}{8}p^3y^{-1}$$

Here $m=\frac{1}{2},\ k=-\frac{1}{8},\ a=-1,\ b=3.$

$\therefore$ $(m-1)\ (b-1)=(\frac{1}{2}-1)\ (3-1)=-1=a$

Since the condition (3) is satisfied, the primitive can be obtained by substituting the values of a, b, m, k in (4)

$$\therefore \quad y^{\frac{1}{2}}=\mathrm{A}x-\mathrm{A}^3=\mathrm{A}(x-\mathrm{A}^2)$$

or $$y=\mathrm{A}^2(x-\mathrm{A}^2)^2.$$

(ii) Consider the equation

$$p^4=4y(xp-2y)^2.$$

This is written $y=\frac{1}{2}xp-\frac{1}{4}p^2y^{-\frac{1}{2}}.$

Here $m=\frac{1}{2},\ k=-\frac{1}{4},\ a=-\frac{1}{2},\ b=2.$

$\therefore$ $(m-1)(b-1)=(\frac{1}{2}-1)(2-1)=-\frac{1}{2}=a.$

The primitive is therefore

$$y^{\frac{1}{2}}=\mathrm{A}x-\tfrac{1}{4}(2\ \mathrm{A})^2$$

i.e., $$y=\mathrm{A}^2(x-\mathrm{A})^2.$$

2. Next, consider the equation

$$y=mxp+kx^ap^b.$$

This can be put in the form

$$x=\frac{1}{m}\,\frac{y}{p}-\frac{k}{m}x^ap^{b-1};$$

or $$=\frac{1}{m}\,y\left(\frac{1}{p}\right)-\frac{k}{m}x^a\left(\frac{1}{p}\right)^{1-b}.$$

Suppose x is the dependent and y the independent variable, and the equation is of the form considered in § 1. Hence it is capable of reduction to Clairaut's form, if

$$\left(\frac{1}{m}-1\right)(1-b)=a,$$

i.e., $$(m-1)\ b=am \qquad \ldots \qquad \ldots \qquad \ldots \quad (5)$$

If this condition is satisfied, the primitive can be obtained in the form

$$x^m=\mathrm{A}y-\frac{k}{m}\,(\mathrm{A}/m)^{1-b} \qquad \ldots \qquad \ldots \quad (6)$$

Examples :—

(1) $y=-xp+x^4p^2$.

Here $m=-1,\ k=1,\ a=4,\ b=2.$

$\therefore$ $(m-1)b=(-1-1)2=-4=am$

Hence the primitive is

$$x^{-1}=Ay+(-A)^{-1}.$$

$\therefore$ $A^2xy=x+A.$

(2) $x^3p^2+x^2yp+d^3=0.$

Here we have $y=-xp-d^3x^{-2}p^{-1}$,

and $m=-1,\ k=-d^3,\ a=-2,\ b=-1.$

$\therefore$ $(m-1)\ b=(-2)(-1)=2=am.$

Hence the primitive is

$$x^{-1}=Ay-d^3(-A)^2,$$

i.e., $A^2d^3x-Axy+1=0.$

J. C. Swaminarayan.

On the Face of the Sky for January and February 1913.

(The time referred to is Standard time).

Sidereal time at 8 p.m.

	January. H.	M.	S.	*February.* H.	M.	S.
1	2	42	32	4	44	45
8	3	10	8	5	12	21
15	3	37	44	5	39	57
22	4	5	20	6	7	33
29	4	32	56	6	35	9

From this table the constellations visible during the evenings can be ascertained by a reference to their position as given in the star-charts.

The Planets.

Mercury which is a morning star in direct motion at the beginning of the year, will be at superior conjunction on Feb. 1913, when it becomes an evening star. It is in conjunction with the moon on January 5 and on February 5, with Mars on January 9, and with Jupiter on Jan. 11.

Venus which is evening star in the beginning of the year continues to be so in these months.

Mars is a morning star in these months. It is in conjunction with the moon on February 3, and with Jupiter on January 12.

Jupiter, also is a morning star in these months. It is in conjunction with the moon on February 2.

Saturn which is in retrograde motion at the beginning of the year is stationary on January 29, when it resumes direct motion. It is in conjunction with the moon on February 14 at 9·3 P.M.

V. RAMESAM.

REVIEWS.

Cours de Mathématiques Supérieures à l' Usage des Candidats à la Licence ès Sciences Physiques, by M. l' Abbè E. Stoffaes Professor of the Catholic Science College and Director of the Catholic Institute of Applied Science and Art of Lille. Third edition, entirely recast. Paris, Gauthier-Villars, 1911. Two volumes ; 10 francs each.

In the introduction, the author explains the purpose of his work. "What is required of students of physics is a thorough acquaintance with the chief investigations in mathematics and with the common methods of calculation. The first edition of the work dates from 1891. Since then, most Colleges have provided for a Certificate of General Mathematics, and in several Universities there have been established special courses arranged to suit Industry and the Applied Sciences. In a lesson book, I have taken no pains to be original.......Engaged for nearly ten years in teaching mathematics at the Catholic Institute of Applied Science and Art at Lille, I have had opportunity to learn how rapidly the level of theoretical studies has risen in such schools and from experience I can say that few items could have been omitted from this book."

The first volume begins with a synopsis of the more elementary parts of *higher algebra* ; the remainder theorem, determinants, the index law, permutations and combinations, imaginaries, limits, series, and logarithms are dealt with in some eighty pages. The treatment is simple and clear, and no attempt is made to seek after novelty ; it is the highroad that is followed. The treatment of series, however, deserves special mention ; and the proof that $(1+1/m)^m$ tends to e as m approaches infinity is a great improvement on the usual proofs in brevity and accuracy. The device adopted is to put

$$(1-1/m)(1-2/m)\ldots\ldots\ldots(1-\overline{n-1}/m)=1-n(n-1)/2m.\,\theta,$$

where θ lies between 0 and 1. It readily follows that $(1+1/m)^m$ falls short of e by less than $e/2m$. This is, of course, when m is a positive integer ; and the necessary extensions are easily made.

Then follow the elements of the differential calculus, the theory of equations and plane and solid analytical geometry. These seem somewhat ineffective, as they represent what can be learnt without a teacher. The last part of the volume amounting to one-third of it and dealing with the calculus, differential and integral, however, deserves warm

commendation, though the rigid adherence to the highway method noted already, makes it difficult to select anything special to remark on.

Except for a short account of the differential equations, the second volume is devoted to more serious analytical geometry, free use being made of the calculus. This is not so elementary as the rest of the book.

In a work designed for students of physics, one is surprised to find that there is no mention of even the more elementary parts of the apparatus for solving Laplace's Equation.

The reader is not in a position to know what exactly is the use of a good deal that is contained in the second volume. Perhaps the author will bring out a third volume of about equal size dealing with Laplace's Equation. The subject has been well treated in the first part of Webster's Electricity and Magnetism; but that writer has not our author's instinct for the straight-forward method.

In accordance with the practice of continental writers, only a few examples are included. The absence of numerous examples, reduces the work to a reasonable compass, and makes it easier for the reader to appreciate the logical development of the subject. Accordingly, a student entering upon a course in "higher mathematics" would, after going through this book, feel reassured as to the task before him. The clearness and orthodoxy of treatment render the work admirable as a College Manual.

The Ganita-Sara-Sangraha of Mahaviracharya, with English Translation and Notes: by *Rao Bahadur* Professor M. Rangacharya M.A. (Government Press, Madras).

This is a work of considerable historical value and the editor has placed the mathematical world under a deep debt of obligation by his publication.

Hitherto very few Indian scholars have worked in the field of unearthing the ancient mathematical material known to exist in India. This may be due "to the fact that it is hard to find a man with the requisite scholarship, who can afford to give his time to what is necessarily a labour of love." It is, therefore, with pride and gratification that we welcome the appearance of such a man in South India, whose disinterested labours have thrown a new light upon the subject of Oriental Mathematics.

Further, we have been accustomed to think of Northern and Wesern India as "the ancient habitats of Hindu Mathematics" and the labours of great European Scholars like Colebrooke and Taylor were confined to those parts of India; this work proves that som schools of mathematics had existed in South India also.

The author of this ancient work, Mahaviracharya, appears to have lived in the court of a monarch who ruled over much of what is now the Province of Mysore, in the first half of the 9th century A. D., and the date of the treatise under review may be roughly taken as 850 A. D. Thus Mahaviracharya comes between Brahmagupta of the 7th century and Bhaskara of the 12th century, whose works are well known. Hence the historical importance of this publication.

Professor D. E. Smith of the Columbia University, New York, has written a thoughtful and instructive introduction to this book in which he discusses various questions as regards ancient Hindu Mathematics, in the light of this publication. We greatly commend this iotroductfon to our readers.The following concluding sentences may be of interest :—

"Professor Rangacharya has, by his great contribution to the history of Mathematics, confirmed the view, already taking rather concrete form, that India developed an algebra of her own;.........that India influenced Europe in the matter of algebra, more than she was influenced in return; that there was no native geometry really worthy of the name; that trigonometry was practically non-existent save as imported from the Greek astronomers; and that whatever of geometry was developed came probably from Mesopotamia rather than from Greece....... His labours have given us further evidence of the fact that Oriental Mathematics lacks the cold logic, the consecutive arrangement and the abstract character of Greek Mathematics, but it possesses a richness of imagination, an interest in problem-setting, and poetry, all of which are lacking in the treatises of the West..........."

Coming to the body of the work, it has nine chapters. In the first chapter are given the names of notational places and the rules for arithmetical operations with zero, positive and negative quantities. The decimal scale of notation is used and there are names given for the first 24 places. In Chapter II are explained the arithmetical operations of multiplication, division, squaring, cubing, extraction of square root and cube root and summation of series in A. P. and G. P. Chapters II and IV deal with fractions and problems thereon. Chapter V treats of problems on the 'rule of three.' Chapter VI deals with problems on the

double rule of three, interest, propotionate division and indeterminate equations of the first degree. The next chapter deals with the mensuration of areas, such as triangles, quadrilaterals, circles, and curvilinear areas. Here the value of π given is $\sqrt{10}$ Chapter VIII deals with the cubical contents of solids *i.e.*, calculations relating to excavations, sawing of logs of wood, piles of bricks, &c. Chapter IX treats of shadows and problems relating thereto.

The original sanskrit is written in too terse and concise a style for an ordinary scholar to understand the exact meanings of the slokas. If we add to this, the difficulty of expounding technical terms and phrases, and the labour involved in the task of collecting and deciphering manuscripts, the editor may well be congratulated on the success with which he has brought out this publication.

P. V. S.

SOLUTIONS.

Question 289.

(S. RAMANUJAN) :—Find the value of

(i) $1+\sqrt{\{1+2\sqrt{[1+3\sqrt{(1+\&c\ldots\ldots)}]}\}}$

(ii) $1+\sqrt{\{6+2\sqrt{[7+3\sqrt{(8+\&c\ldots\ldots)}]}\}}$.

Solution by the Proposer.

(i) $n(n+2)=n\sqrt{1+(n+1)(n+3)}$

Let $n(n+2)=f(n)$, then we see that

$$f(n)=n\sqrt{1+f(n+1)}=n\sqrt{1+(n+1)\sqrt{1+f(n+2)}},$$

$$=n\sqrt{1+(n+1)\sqrt{1+(n+2)\sqrt{1+f(n+3)}}},\text{ and so on ;}$$

that is $n(n+2)=n\sqrt{1+(n+1)\sqrt{1+(n+2)\sqrt{1+(n+3)\sqrt{1+\&c.}}}}$

Putting $n=1$ we have

$$1\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{1+\&c}}}}}=3.$$

(ii) In a similar manner,

$$n(n+3)=n\sqrt{n+5+(n+1)(n+4)}.$$

Supposing $f(n)=n(n+3)$, we have,

$$f(n)=n\sqrt{n+5+f(n+1)}=n\sqrt{n+5+(n+1)\sqrt{n+6+f(n+2)}},$$

$$=n\sqrt{n+5+f(n+1)\sqrt{n+6+(n+2)\sqrt{n+7+f(n+3)}}},\text{ and so on.}$$

Thus $n(n+3)=n\sqrt{(n+5)+(n+1)\sqrt{n+6+(n+2)\sqrt{n+7+\&c}}}$

Hence $$1\sqrt{6+2\sqrt{7+3\sqrt{8+4\sqrt{9+5\sqrt{10+\&c}}}}}=4.$$

Question 323.

(J. C. SWAMINARAYAN M.A.) :—Shew that, with the usual notation of a plane triangle, the determinant

$$\begin{vmatrix} b^2f^2, & c^2f^2, & 1 \\ (ch+af)^2, & c^2g^2, & 1 \\ b^2p^2, & (af+bg)^2, & 1 \end{vmatrix} = 2abcf(af+bg+ch)\,[gh+hf\cos C+fg\cos B-f^2\cos A].$$

Solution by the Proposer.

The given determinant is equal to

$$\begin{vmatrix} f^2, & f^2, & 1 \\ (ch+af)^2, & b^2g^2, & b^2 \\ c^2h^2, & (af+bg)^2, & c^2 \end{vmatrix}$$

$$=(af+bg+ch)\begin{vmatrix} o, & f^2, & 1 \\ ch+af-bg, & b^2g^2, & b^2 \\ ch-af-bg, & (af+bg)^2, & c^2 \end{vmatrix}$$

$$=2\,a(af+bg+ch)\begin{vmatrix} o, & f^2, & 1 \\ ch+af-bg, & b^2g^2, & b^2 \\ -f, & bfg, & -b\cos C \end{vmatrix}$$

$$=2a(af+bg+ch)\begin{vmatrix} o, & f^2, & 1 \\ ch+af-bg, & bg(ch+af), & b^2 \\ -f, & o, & -b\cos C \end{vmatrix}$$

$$=2abf(af+bg+ch)\begin{vmatrix} o, & f, & 1 \\ ch+af-bg, & g(ch+af), & bf \\ -1, & o, & -\cos C \end{vmatrix}$$

$$=2abf(af+bg+ch)[f(ch+af-bg)\cos C-b^2b+gch+aby]$$

$$=2abf(af+bg+ch)[cgh+hfc\cos C+fg(a-b\cos C)-f^2(b-a\cos C)]$$

$$=2abcf(af+bg+ch)[gh+hf\cos C+fg\cos B-f^2\cos A.]$$

Question 333.

(SELECTED) :—A school master went mad and amused himself by arranging the boys. He turned the *dux* boy down one place, the next *dux* two places, the next three, and so on, until every boy's place had been altered at least once. Then he began again and so on: till after 306 turnings down all the boys got back to their original places. This disgusted him and he kicked one boy out. Then he was amazed to find that he had to operate 1120 times before all got back to their original places. How many boys were in the class?

Solution by A. C. L. Wilkinson M.A., F.R.A.S.

If there are $n-1$ boys in the class, we require $n-1$ turnings down so as to alter every boys place. We thus obtain a substitution which by continued repetition gives the original order. We expect therefore that 306 and 1120 should be devisable by two consecutive integers and these are 17 and 16. There were 18 boys in the class.

The substitution for 17 boys is in fact

$$\begin{pmatrix} 5. & 10. & 3. & 11. & 6. & 12. & 2. & 13. & 7. & 14. & 4. & 15. & 8. & 16. & 1. & 17. & 9 \\ 1. & 2. & 3. & 4. & 5. & 6. & 7. & 8. & 9. & 10. & 11. & 12. & 13. & 14. & 15. & 16. & 17 \end{pmatrix}$$

which is compounded of the circular substitutions

$$(1.\ 15.\ 12.\ 6.\ 5)(2.\ 7.\ 9.\ 17.\ 16.\ 14.\ 10)(3)(4.\ 11)(8.\ 13)$$

and the L.C.M. of 5, 7, 1, 2, 2 is 70. Thus the number of turnings down required is $16\times70=1120$.

Question 363.

(SELECTED):—Give a proof of the correctness of the Russian peasant method of multiplication, which may be described as follows:

To multiply a by b write down $a\times b$. Under a write the exact or lower quotient obtained by dividing it by 2; under this quotient write the quotient obtained by dividing it by 2; and so on, until we obtain the quotient 1.

Under b write its double, under this double its double, and so on, until we have as many numbers in the second column as in the first.

Next, add the numbers in the second column which correspond to odd numbers in the first. The result is the product of a and b.

Remarks by 'Zero' 0,.

The proof depends on the theorems given on pages 167, 168 in Part I of Chrystal's *Text-Book of Algebra*, and is quite simple.

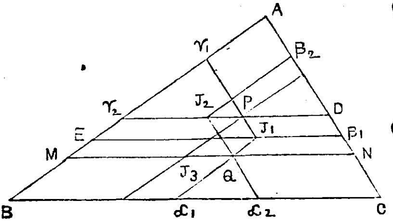
Question 365.

(T. SUNDARA RAO):—The Jerabek points J_1, J_2 of a $\triangle$ ABC are such that their distances from the sides of the $\triangle$, measured parallel to (AB, BC, CA) and (BA, AC, CB) respectively, are all equal (J.I.M.S, Vol IV, p. 18). Shew that associated with these points is a third point J_3, such that if parallels are drawn from it to the sides of the $\triangle$, the parallels drawn *in one sense* are each equal to the parallels drawn *in the opposite sense* from J_1, J_2.

Solution by V. V. Satyanarayana.

Let $J_1\alpha_1$ and $J_1\gamma_1$ intersect $J_2\alpha_2$ and $J_2\gamma_2$, produced if necessary, in Q and P respectively. Through P and Q draw parallels to AB and BC respectively to intersect at J_3. Then shall J_3 be the required point.

For, let J_3Q cut AB, AC in M, N respectively; and let $J_1\beta_1$, $J_2\gamma$ cut AB, AC in E and D respectively. Then



(1) $J_3M = P\gamma_2 = PJ_2 + J_2\gamma$
$= PJ_2 + J_1\beta_1$,
since $J_2\gamma_2 = J_1\beta_1$,
$= PJ_2 + PD$
$= J_2D$.

(2) $J_3N = J_3Q + QN$
$= J_3Q + J_2D$
$= J_3Q + J_3M$
$= QM = J_1E$.

By drawing all the parallels and equating the various parallels required, the other results follow.

Remarks by the Proposer.

J_3 is the isotomic conjugate of the incentre of the triangle.

The normal of cordinates of J_1, J_2, J_3 are b, c, a ; c, a, b ; $1/a^2$, $1/b^2$, $1/c^2$ and the barycentric cordinates are

$$1/c,\ 1/a,\ 1/b\ ;\ 1/b,\ 1/c,\ 1/a\ ;\ 1/a,\ 1/b,\ 1/c.$$

Question 382.

(V. Ramaswami Aiyar, M.A.) :—ABC, DEF are two trirangles self conjugate with respect to a rectangular hyperbola. Prove that the pedal circles of the points A,B,C with respect to the triangle DEF, and the pedal circles of the points D,E,F with respect to the triangle ABC are all coaxal.

Solution by M. Bhimasena Rao.

Let D′, E′, F′ be the isogonal conjugates of D,E,F with respect to the triangle ABC. The polar of any point with respect to a rect. hyp. self-conjugate with respect to ABC passes through the isogonal conjugate of the point. Since DEF is self-conjugate with respect to the given rect. hyp., as also ABC, it follows that D′, E′, F′ lie respectively on EF, FD, DE. Further, as ABC, DEF are both self-conjugate triangles, a conic goes round them. Therefore D′,E′,F′ are on a line which is the isogonal transformation of the conic round ABC, DEF.

Let d, e, f; d', e', f', be the feet of the perpendiculars from D, F, F, D′E′, F′, on any side of the triangle ABC. Then by projection

$$\frac{FD \cdot FD'}{FE \cdot FE'} = \frac{fd . fd'}{fe \, fe'} = \frac{\text{power of } f \text{ with regard to pedal} \odot \text{ of D}}{\text{power of } f \text{ with regard to pedal} \odot \text{ of E}}.$$

In other words, the ratio of the powers of the foot of the perpendiculars from F on any side of the triangle ABC, with respect to the pedal circles of D and E is the same. Therefore, the pedal circles of D, E, F are coaxal.

Similarly, the pedal circles of A, B, C with respect to DEF are co-axal; and they belong to the same co-axal-system by Question 192 of the Proposer.

Question 383.

(D. D. KAPADIA, M.A., BSc,):—If n is a positive integer, shew that

$$\tfrac{1}{2}\sin n\alpha + \sin(n-1)\alpha\cos\theta + \sin(n-2)\alpha\cos 2\theta$$
$$+ \ldots\ldots + \sin\alpha\cos(n-1)\theta = \tfrac{1}{2}\frac{\cos n\alpha - \cos n\theta}{\cos\alpha - \cos\theta}\sin\alpha.$$

Hence or otherwise deduce the results :—

(i) $(n-1)\cos\theta + (n-2)\cos 2\theta + (n-3)\cos 3\theta + \ldots\ldots + \cos(n-1)\theta$

$$= \tfrac{1}{2}\left(\sin^2\frac{n\theta}{2}\operatorname{cosec}^2\frac{\theta}{2} - n\right);$$

(ii) $(n-1)^2\sin\theta + (n-2)^2\sin 2\theta + \ldots\ldots + \sin(n-1)\theta$

$$= \tfrac{1}{4}\sin\theta\operatorname{cosec}^2\frac{\theta}{2}\left(n^2 - \operatorname{cosec}^2\frac{\theta}{2}\sin^2\frac{n\theta}{2}\right)$$

(iii) $(n-1)^3\cos\theta + (n-2)^3\cos 2\theta + (n-3)^3\cos 3\theta + \ldots\ldots + \cos(n-1)\theta.$

$$= \tfrac{1}{4}\left(3n^2\operatorname{cosec}^2\frac{\theta}{2} - 3\sin^2\frac{n\theta}{2}\operatorname{cosec}^4\frac{\theta}{2} - 2n^3\right) + \tfrac{1}{2}\sin^2\frac{n\theta}{2}\operatorname{cosec}^2\frac{\theta}{2}$$

What is the sum of the series

$$(n-1)^4\sin\theta + (n-2)^4\sin 2\theta + \ldots\ldots + \sin(n-1)\theta.$$

Solution by K. J. Sanjana, M.A., T. P. Trivedi, M.A., LL.B., A. A. Krishnaswami Aiyangar and G. Ramachandran.

The first result is the well-known identity of Hermite, and is given by Hobson. Both the identity and result (iii) were set in the Mathematical Tripos, Part II, 1911.

Multiply the left side by 2 $(\cos\alpha - \cos\theta)$, and split up each term, keeping $\cos\theta$, $\cos 2\theta$, $\cos 3\theta$...unchanged; we shall get, on cancelling out,

$\sin n\alpha \cos\alpha - \cos n\theta \sin\alpha - \sin(n-1)\alpha$

$$=(\cos n\alpha - \cos n\theta)\sin\alpha,$$

which proves the result.

(i) Divide out by $\sin\alpha$, and put $\alpha=0$; we get

$$\tfrac{1}{2}n+(n-1)\cos\theta+(n-2)\cos 2\theta \ldots +2\cos(n-2)\theta+\cos(n-1)\theta$$

$$=\tfrac{1}{2}\frac{1-\cos n\theta}{1-\cos\theta}=\tfrac{1}{2}\sin^2\tfrac{1}{2}n\theta.\operatorname{cosec}^2\tfrac{1}{2}\theta$$

(ii) Differentiate the identity for θ; then

$$-\sin\alpha.(n-1)\sin(n-1)\theta-\sin 2\alpha.(n-2)\sin(n-2)\theta \ \ldots$$

$$-\sin(n-2)\alpha.\,2\sin 2\theta-\sin(n-1)\alpha.\sin\theta=\tfrac{1}{2}\frac{n\sin n\theta\sin\alpha}{\cos\alpha-\cos\theta}$$

$$-\tfrac{1}{2}\frac{\cos n\alpha-\cos n\theta}{(\cos\alpha-\cos\theta)^2}\sin\alpha\sin\theta.$$

Divide out by $\sin\theta$, and put $\theta=0$; we get

$$-(n-1)^2\sin\alpha-(n-2)^2\sin 2\alpha \ldots -\sin(n-1)\alpha$$

$$=\tfrac{1}{2}\frac{n^2\sin\alpha}{\cos\alpha-1}-\tfrac{1}{2}\frac{\sin\alpha(\cos n\alpha-1)}{(\cos\alpha-1)^2}$$

$$=\tfrac{1}{2}\cot\frac{\alpha}{2}\left(\sin^2\frac{n\alpha}{2}\operatorname{cosec}^2\frac{\alpha}{2}-n^2\right).$$

Change signs and put θ for α.

(iii) Differentiate the identity twice for α; the left side gives

$$-\tfrac{1}{2}n^2\sin n\alpha-(n-1)^2\sin(n-1)\alpha\cos\theta-$$

$$(n-2)^2\sin(n-2)\alpha\cos 2\theta \ldots -\sin\alpha\cos(n-1)\theta.$$

Divide by $\sin\alpha$, and put $\alpha=0$; we get

$$-\tfrac{1}{2}n^3-\{(n-1)^3\cos\theta+(n-2)^3\cos 2\theta \ldots +1^3.\cos(n-1)\theta\}.$$

Call the right side y, multiply by $2(\cos\alpha-\cos\theta)$, and differentiate twice; we get

$$2\frac{d^2y}{d\alpha^2}(\cos\alpha-\cos\theta)=4\frac{dy}{d\alpha}\sin\alpha+2y\cos\alpha-n^2\cos n\alpha\sin\alpha-2n\sin n\alpha\cos\alpha$$

$$-\sin\alpha(\cos n\alpha-\cos n\theta).$$

Hence $\left[\dfrac{d^2y}{d\alpha^2}\div\sin\alpha\right]=\dfrac{1}{2(\cos\alpha-\cos\theta)}\times$

$$\left\{4\frac{dy}{d\alpha}+2\left(\frac{y}{\sin\alpha}\right)\cos\alpha-n^2\cos n\alpha-2n\left(\frac{\sin n\alpha}{\sin\alpha}\right)\cos\alpha\right.$$

$$\left.-(\cos n\alpha-\cos n\theta)\right\};$$

putting $\alpha=0$, we get

$$\tfrac{1}{4}\operatorname{cosec}^2\frac{\theta}{2}\left\{2\sin^2\frac{n\theta}{2}\operatorname{cosec}^2\frac{\theta}{2}+\sin^2\frac{n\theta}{2}\operatorname{cosec}^2\frac{\theta}{2}-n^2-2n^2-2\sin^2\frac{n\theta}{2}\right\}$$

$$=\frac{3}{4}\sin^2\frac{n\theta}{2}\operatorname{cosec}^4\frac{\theta}{2}-\frac{3}{4}n^2\operatorname{cosec}^2\frac{\theta}{2}-\frac{1}{2}\sin^2\frac{n\theta}{2}\operatorname{cosec}^2\frac{\theta}{2},$$

which gives the result.

Differentiating thrice for θ, the left side gives

$$\sin\alpha.\,(n-1)^3\sin(n-1)\theta+\sin 2\theta.\,(n-2)^3\sin(n-2)\theta\ldots\ldots+\sin(n-1)\alpha.\,\sin\theta;$$

Divide out by $\sin\theta$, and put $\theta=0$. We obtain

$$(n-1)^4\sin\alpha+(n-2)^4\sin 2\alpha\ldots\ldots\ldots\ldots+1^4\sin(n-1)\alpha.$$

On the right side we obtain

$$\left(\frac{d^3y}{d\theta^3}\div\sin\theta\right)_0=\tfrac{1}{4}\operatorname{cosec}^2\frac{\alpha}{2}\sin\alpha\left\{n^4-3n^2\operatorname{cosec}^2\frac{\alpha}{2}-\sin^2\frac{n\alpha}{2}\operatorname{cosec}^2\frac{\alpha}{2}+3\sin^2\frac{n\alpha}{2}\operatorname{cosec}^4\frac{\alpha}{2}\right\},$$

after some reduction.

Put θ for α and the required sum is obtained.

Question 385.

(SELECTED):—If $s_r\equiv 1^r+2^r+\ldots\ldots\ldots+n^r$, shew that $8\,(s_3+s_5+s_7)+1$ is a perfect square.

Solution (1) *by N. Durai Rajan*, (2) *by N. P. Pandya, Keshavlal D. Shah, and G. Ramachandran*; (3) *by J. C. Swaminarayan, M.A., T. P. Trivedi and N. Sanjiva Rao.*

(1) Let $t_n=(n+1)^4n^4-n^4(n-1)^4=8n^7+8n^5$. Then

$$\Sigma(t_n)=(n+1)^4n^4=8(s_7+s_5);$$

and $$4s_3=n^2(n+1)^2.$$

$$\therefore\quad 8(s_7+s_5+s_3)+1=16s_3^2+8s_3+1=[n^2(n+1)^2+1]^2$$

(2) Let $n(n+1)=2p$. Then we have

$$s_3\equiv 1^3+2^3+\ldots n^3=p^2,$$
$$s_5\equiv 1^5+2^5+\ldots n^5=\tfrac{1}{3}p^2(4p-1),$$
$$s_7\equiv 1^7+2^7+\ldots n^7=\tfrac{1}{3}p^2(6p^2-4p+1),$$

as is easily seen.

$$\therefore\quad 8(s_3+s_5+s_7)+1=8p^2\left\{1+\tfrac{1}{3}(4p-1)+\tfrac{1}{3}(6p^2-4p+1)\right\}+1,$$
$$=8p^2(2p^2+1)+1,$$
$$=16p^4+8p^2+1,$$
$$=(4p^2+1)^2,$$
$$=\left\{(n^2+n)^2+1\right\}^2.$$

(3) Generally

$$(n+1)^{r+1}-(n+1)={}_{r+1}c_1.s_r+{}_{r+1}c_2.s_{r-1}+\dots+{}_{r+1}c_r.s_1$$

$$\begin{aligned}
\text{Whence } (n+1)^8-(n+1) &= 8s_7+28s_6+56s_5+70s_4+56s_3+28s_2+8s_1\\
-4\{(n+1)^7-(n+1)\} &= -28s_6-84s_5-140s_4-140s_3-84s_2-28s_1\\
+6\{(n+1)^6-(n+1)\} &= +36s_5+90s_4+120s_3+90s_2+36s_1\\
-4\{(n+1)^5-(n+1)\} &= -20s_4-40s_3-40s_2-20s_1\\
+3\{(n+1)^4-(n+1)\} &= +12s_3+18s_2+12s_1\\
-4\{(n+1^3)-(n+1)\} &= -12s_2-12s_1\\
+2\{(n+1)^2-(n+1)\} &= +4s_1\\
+1 &= +1.
\end{aligned}$$

$$\therefore\quad (n+1)^8-4(n+1)^7+6(n+1)^6-4(n+1)^5+3(n+1)^4$$
$$-4(n+1)^3+2(n+1)^2+1=8(s_7+s_5+s_3)+1,$$

i.e. $\{(n+1)^4-2(n+1)^3+(n+1)^2+1\}^2=8(s_7+s_5+s_3)+1.$

Question 388.

(M. V. ARUNACHALA SASTRY M.A.) :—If $al+bm+cn=(a+b+c)p$, prove that $al^2+bm^2+cn^2>(a+b+c)p^2$; a, b, c being positive quantities.

Solution by T. P. Trivedi M.A., L.L.B, H. V. Venkataramiengar, and several others.

We have to prove that

$$al^2+bm^2+cn^2>(a+b+c)p^2$$
$$>\frac{(a+b+c)(al+bm+cn)^2}{(a+b+c)^2};$$

or $$(al^2+bm^2+cn^2)(a+b+c)-(al+bm+cn)^2>0,$$

or $$ab(l-m)^2+bc(m-n)^2+ca(n-l)^2>0,$$

which is the case.

Question 389.

(D. D. KAPADIA, M.A., B. Sc.) :—A square $A_1\ A_2\ A_3\ A_4$ and a regular pentagon $B_1\ B_2\ B_3\ B_4\ B_5$ are inscribed in a circle of radius r. Shew that the continued product of all the chords AB is numerically equal to $2r^{20}\sin 20\,\theta$, where 2θ is the angle subtended by any one of these chords at the centre of the circle.

Solution by J. C. Swaminarayan, M.A., B. P. Mathur, T. P. Trivedi, M.A., N. Sanjiva Rao, and N. B. Pendse.

We have the following general formula

$$\sin n\theta=2^{n-1}\sin\theta\sin\left(\theta+\frac{\pi}{n}\right)\sin\left(\theta+\frac{2\pi}{n}\right)\ \dots\ \sin\left\{\theta+\frac{(n-1)\pi}{n}\right\}$$

By hypothesis, θ is the angle subtended by A_1B_1 at the circumference of the circle, and a side of a regular pentagon subtends an angle equal to $\frac{\pi}{5}$ at the circumference.

$\therefore\quad A_1B_1.A_1B_2.A_1B_3.\ A_1B_4.A_1B_5$

$$=2^5r^5\sin\theta\sin\left(\theta+\frac{\pi}{5}\right)\sin\left(\theta+\frac{2\pi}{5}\right)\sin\left(\theta+\frac{3\pi}{5}\right)\sin\left(\theta+\frac{4\pi}{5}\right),$$
$$=2r^5\sin 5\theta.$$

Hence, the continued product of all the chords AB is

$$2^4r^{20}\sin 5\theta\sin\left(5\theta+\frac{\pi}{4}\right)\sin\left(5\theta+\frac{2\pi}{4}\right)\sin\left(5\theta+\frac{3\pi}{4}\right)$$

which is equal to $2r^{20}\sin 20\theta$.

Question 394.

(T. P. Trivedi, M.A., LL. B.) :—In a spherical triangle, if

$$x=\cot\frac{b}{2}\cot\frac{c}{2},\ y=\cot\frac{c}{2}\cot\frac{a}{2},\ z=\cot\frac{a}{2}\cot\frac{b}{2},$$

and
$$X=\tan\frac{B}{2}\tan\frac{C}{2},\ Y=\tan\frac{C}{2}\tan\frac{A}{2},\ Z=\tan\frac{A}{2}\tan\frac{B}{2},$$

then
$$(1-x)(Y-Z)=(1-X)(y-z),$$

with two similar equalities.

Solution by G. Ramachandran, and others.

We have

$$\frac{1-X}{y-z}=\frac{1-\tan\frac{B}{2}\tan\frac{C}{2}}{\tan\frac{A}{2}\left(\tan\frac{C}{2}-\tan\frac{B}{2}\right)}=\frac{\cos\frac{B+C}{2}}{\tan\frac{A}{2}\sin\frac{C-B}{2}}$$

$$=\frac{\cos\frac{B+C}{2}}{\sin\frac{A}{2}}\div\frac{\sin\frac{C-B}{2}}{\cos\frac{A}{2}},$$

$$=\frac{\cos\frac{b+c}{2}}{\cos\frac{a}{2}}\div\frac{\sin\frac{c-b}{2}}{\sin\frac{a}{2}}, \text{by Delambre's Analogies,}$$

$$=\frac{\cos\frac{b+c}{2}}{\cot\frac{a}{2}\sin\frac{c-b}{2}}=\frac{1-\cot\frac{b}{2}\cot\frac{c}{2}}{\cot\frac{a}{2}\left(\cot\frac{c}{2}-\cot\frac{b}{2}\right)}=\frac{1-x}{y-z}$$

Thus

$$(1-x)(Y-Z)=(1-Z)(y-z).$$

Question 395.

(K. J. SANJANA, M.A.) :—Prove the identity

$$\Sigma a(b-c)(b+c-a)(b^p+c^p-a^p) = -2\Sigma a^p bc\,(b-c)$$
$$= 2abc(b-c)(c-a)(a-b)\,S_{p-3}$$

when S_{p-3} stands for the sum of the homogeneous products of $(p-3)$ dimensions: and find when the left hand side identically vanishes.

(Suggested by Q. 351 of Prof. J. C. Swaminarayan.)

Solution by T. P. Trivedi, M.A ; L.L.B.; J. C. Swaminarayan, M.A., and N. Sanjeva Rao.

We have

$$a(b-c)(b+c-a)(b^p+c^p-a^p)$$
$$= \{ b^{p+2}a - c^{p+2}a + (b-c)a^{p+2} \} - \{ a^2b^{p+1} - a^2c^{p+1}$$
$$+ a^{p+1}(b^2-c^2) \} - b^p ac(c-a) - c^p ab(a-b).$$

On adding two similar terms, the left-side is clearly seen to be

$$-2\Sigma a^p bc(b-c).$$

This is the same as

$= 2abc(a-b)(b-c)(c-a)\,S_{p-3}$, by a well known formula.

It is easily seen that the expression will vanish if $p=0$, 1 or 2.

Question 397.

(J. C. SWAMINARAYAN, M.A.) :—A particle moves under an attractive force $\dfrac{\mu m r}{(r^2+k^2)^2}$, and is projected from an apse at a distance a with a velocity equal to that acquired by falling from infinity to the apse under the action of the same force. Shew that it will describe a circle of radius $\frac{1}{2}(a+k^2/a)$ with its centre at a distance $\frac{1}{2}(a-k^2/a)$ from the centre of force.

Solution by M. K. Sadasiva Aiyar, N. B. Pendse, and the Proposer.

$$F = \frac{\mu m r}{(r^3+k^2)^3} \;\therefore v^2 = c - \int \frac{2\mu m r\,dr}{(r^2+k^2)^3}$$
$$= \frac{\mu m}{(r^2+k^2)^2}, \text{ since } c=0.$$

But
$$v = \frac{h}{p}\;;\;\therefore\;\frac{h}{p} = \frac{\sqrt{\mu m}}{(r^2+k^2)}.$$

At the commencement $p=r=a$;

$$\therefore\quad \frac{h}{a} = \frac{\sqrt{\mu m}}{a^2+k^2}$$

The equation of the locus is, therefore,

$$r^2+k^2 = (a^2+k^2)\,p/a \qquad \ldots \quad (1)$$

The pedal equation of a circle of radius b, centre distant c from origin is $2\,bp=r^2+b^2-c^2$ ... ... (2)

Comparing (1) and (2), we see, at once, that the locus is a circle of radius $b=\frac{1}{2a}(a^2+k^2)=\frac{1}{2}(a+k^2/a)$.

The distance from the centre of force to the centre of the circle is

$$a-\tfrac{1}{2}(a+k^2/a)=\tfrac{1}{2}(a-k^2/a).$$

Question 399.

(P. V. SESHU AIYAR, B.A., L.T.) :—Solve completely the differential equation :—

$$c\left(x+y\frac{dy}{dx}\right)=y\sqrt{1+\left(\frac{dy}{dx}\right)^2}.$$

Solution by K. J. Sanjana, J. C. Swaminarayan, and others.

Differentiating for x and rearranging, we get

$$(1+p^2+yq)\left\{c-p(1+p^2)^{-\frac{1}{2}}\right\}=0.$$

If $1+p^2+yq=0$, then $1+p^2+y\dfrac{dp}{dy}\dfrac{dy}{dx}=0$,

or $\quad 1+p^2+yp\dfrac{dp}{dy}=0$. Hence $\dfrac{pdp}{1+p^2}=-\dfrac{dy}{y}$,

which gives $1+p^2=k^2y^2$, where k is some constant. Substituting p from this in the given equation, we have

$$c\left\{x+\sqrt{(k^2-y)^2}\right\}=k,$$

that is, $\quad c^2(x^2+y^2)-2ckx-k^2(c^2-1)=0,$ ... ... (1)

which is the general solution.

If $c-p\,(1+p^2)^{-\frac{1}{2}}=0$, then $p=\pm\dfrac{c}{\sqrt{(1-c^2)}}$,

which gives $\quad y=\pm\dfrac{c}{\sqrt{(1-c^2)}}x,$ ... ... (2)

the singular solution.

The *constant-discriminant* of (1) gives

$$4c^2x^2+4c^2(c^2-1)(x^2+y^2)=0$$

or

$$c^2x^2-(1-c^2)y^2=0,$$

the same result as (2). The singular solution, therefore, gives a true envelope. When $c<1$, the system of circles in (1) envelops two straight lines through the origin; when $c=1$, it envelops the axis of y at the origin; when $c>1$, there is no real envelope.

QUESTIONS FOR SOLUTION.

418. (M. Bhimasena Rao):—Tangents at P,Q,R,S on a cubic curve having a centre, cointersect on the curve. Show that the mean centre of P,Q,R,S is the centre of the curve, each asymptote bisecting a pair of opposite connectors of the quadrangle PQRS.

419. (P. A. Subramania Iyer, B.A., L.T.):—Solve completely

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}\left(\frac{1}{y}\frac{dy}{dx}+\frac{2}{x}\right)-\left(\frac{a^2}{2}+\frac{1}{x^2}\right)y=0.$$

420. (K. J. Sanjana, M.A.):—In the plane of three fixed points A,B,C, a fixed straight line is drawn through A, and a variable circle, touches it at that point. Prove that the locus of the intersection of the line joining B to the centre of the circle and the polar of C with regard to the circle is a conic. (Geometrical proof preferred.) Shew that Q.390 is a particular case of this theorem.

If BM,CN are perpendicular to the fixed straight line, examine the locus when AM.AN = − BM.CN.

421. (A. C. L. Wilkinson, M.A., F.R.A.S.):—An ellipsoid of revolution is inscribed in a tetrahedron ABCD with its focus at the centroid G of the tetrahedron. Prove that the plane parallel to BCD which touches the ellipsoid, divides GA in the ratio three times the square of the triangle BCD to four times the sum of the squares o the triangles BCD, CDA, ABC, ABD.

422. (D. D. Kapadia, M.A., B.S.C.):—Shew that—

$$\begin{vmatrix} a, & b, & c, & d, & e, & f \\ f, & a, & b, & c, & d, & e \\ e, & f, & a, & b, & c, & d \\ d, & e, & f, & a, & b, & c, \\ c, & d, & e, & f, & a, & b \\ b, & c, & d, & e, & f, & a \end{vmatrix} = \begin{vmatrix} a+d, & b+e, & c+f \\ c+f, & a+d, & b+e \\ b+e, & c+f, & a+d \end{vmatrix} \begin{vmatrix} a-d, & b-e, & c-f \\ f-c, & a-d, & b-e \\ e-b, & f-c, & a-d \end{vmatrix}$$

423. (D. D. Kapadia, M.A., B.S.C.):—A conic is described having its focus at the origin and osculating the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, at the point whose eccentric angle is α ; show that the directrix of the conic is

$$ax\cos^3\alpha-by\sin^3\alpha+\frac{a^2b^2}{a^2-b^2}=0.$$

424. (T. Rajarama Rao, B.A., B.L.) :—S and S′ are isogonal conjugates of the triangle ABC. SD, SE are drawn perpendicular to AB, AC; and S′D′, S′E′ are drawn perpendicular to the same sides from S′. DE and D′E′ meet at H. The line joining the point H with the centre of the pedal circle meets D′E at Q, and SD′ and S′E, produced meet at P. PA is joined and produced to meet D′E at R. Prove that PR is perpendicular to D′E and also that R,Q,E′,D are concyclic.

425. (R. Srinivasan, M.A.) :—Shew that the envelope of the polar of the focus of the parabola $y^2=4ax$, with respect to any rectangular hyperbola which has four-point contact with it, is a parabola having the same axis as the given parabola.

426. (K. J. Sanjana, M.A.) :—BC is a fixed chord of a given circle and A any point on the arc BAC; P and P′ are isogonally conjugate with regard to the triangle ABC. If the powers of P and P′ with respect to the circle are proportional to their respective distances from BC, prove that the loci of P and P′ are circles, and that AP : AP′ is constant for all positions of A. (Suggested by Q. 393.)

427. (S. Ramanujan) :—Express
$(Ax^2+Bxy+Cy^2)(Ap^2+Bpq+Cq^2)$. in the form $Au^2+Buv+Cv^2$, and hence shew that, if

$$(2x^2+3xy+5y^2)(2p^2+3pq+5q^2)=2u^2+3uv+5v^2,$$

then, one set of the values of u and v, is

$$u=5/2\ (x+y)(p+q)-2xp,\ v=2qy-(x+y)(p+q).$$

428. (S. P. Singaravelu Mudeliar, B.A.) :—Obtain the sextic whose roots are $2\cos\frac{2\pi}{21}$, $2\cos\frac{4\pi}{21}$, $2\cos\frac{8\pi}{21}$, $2\cos\frac{10\pi}{21}$, $2\cos\frac{16\pi}{21}$ and $2\cos\frac{20\pi}{21}$.

429. (S. Narayanan, B.A., L.T.) :—Through any point P, in the plane of a triangle ABC, lines are drawn so that the intercepts made by the arms of each of the three angles may be bisected at P. Shew that their six points of intersection with the three sides lie on a conic X which will be an ellipse, two parallel lines, or a hyperbola, according as P lies *within*, *on*, or *without* the maximum inscribed ellipse of ABC, and that X will be two intersecting lines through P, if P lies on the radical axis of the circumcircle and the polar circle of ABC.

430. (M. BHIMASENA RAO) :—If P is the centre of a conic touching the sides of a triangle ABC at points where the normals are concurrent, show that :

(1) the line joining P to its isogonal conjugate passes through the median point of ABC.

(2) the perpendicular from P on the polar of P with respect to ABC passes through the circumcentre of ABC.

(3) cotPAB+cotPBC+cotPCA=cotPAC+cotPCB+cot PBA.

Hence if a concentric conic be inscribed in the pedal triangle of P, the normals at the points of contact are also concurrent.

431. (P. V. SESHU AYAR, B.A., L.T.) :—Integrate

$$\int \frac{\sin\theta}{\sin^{2-\frac{1}{n}} n\theta}\, d\theta.$$

432. (A. C. L. WILKINSON, M.A., F.R.A.S.) :—The rectangular hyperbolas $x^2-y^2-2\alpha x+2\beta y=16c^2(c^2-\alpha\beta)$, and $xy=c^2$, are so related that triangles can be inscribed in the first that are circumscribed to the second. Show that the locus of the centres of circles circumscribing these triangles is the conic

$$(\beta x+\alpha y)^2=4c^2(xy+\alpha\beta-c^2).$$

433. (J. C. SWAMINARAYAN, M.A.) :—If

$$\begin{aligned} f(x_1, x_2, x_3, \ldots\ldots, x_n) &\equiv p_1(x_1+a_2x_2+a_3x_3+\ldots\ldots+a_nx_n)^2 \\ &+p_2(x_2+b_3x_3+\ldots\ldots+b_nx_n)^2 \\ &+p^2(x_3+c_4x_4+\ldots\ldots+c_nx_n)^2+\ldots\ldots+p_nx_n^2, \end{aligned}$$

shew that

$$\begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2}, & \frac{\partial^2 f}{\partial x_1 \partial x_2}, & \frac{\partial^2 f}{\partial x_1 \partial x_3}, & \ldots\ldots, & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2}, & \frac{\partial^2 f}{\partial x_2^2}, & \frac{\partial^2 f}{\partial x_2 \partial x_3}, & \ldots\ldots, & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & & \ldots\ldots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n}, & \frac{\partial^2 f}{\partial x_2 \partial x_n}, & \frac{\partial^2 f}{\partial x_3 \partial x_n}, & \ldots\ldots, & \frac{\partial^2 f}{\partial x_n^2} \end{vmatrix}$$

is equal to the continued product of $p_1, p_2, p_3, \ldots\ldots p_n$.

434. (COMMUNICATED) :—A railway passenger seated in one corner of the carriage looks out of the window at the further end and observes that a star near the horizon is traversing these windows in the direction of the trains motion and that it is obscured by the partition between

the corner windows on his own side of the carriage and the mid window while the train is moving through the seventh part of a mile. Prove that the train is on the curve, the concavity of which is directed towards the star and which, if it be circular, has a radius of nearly three miles, the breadth of the carriage being 7 ft, and the breadth of the partition 4 inches. [Math. Tripos, 1860].

435. (R. TATA, M.A.) :—If the lines joining the symmedian point of ABC to the middle points of its sides make angles $\theta_1, \theta_2, \theta_3$ with them, shew that

$$\cot\theta_1+\cot\theta_2+\cot\theta_3=0.$$

436. (N. DURAI RAJAN, B.A.) :—Through a point P within a tetrahedron planes are drawn parallel to the four faces. Shew that the twelve points of intersection of these planes and the edges will lie on a sphere, if BC.AD = CA.BD = AB.CD. (This is the analogue of the Lemoine circle).

437. (A. A. KRISHNASWAMI AIYANGAR, B.A.) :—Obtain the following property of Bernoulli's numbers—

$$\frac{B_{4n+1}}{|\underline{4n+2}}\cdot\frac{B_1}{|\underline{2}}+2\cdot\frac{B_{4n-1}}{|\underline{4n}}\cdot\frac{B_3}{|\underline{4}}+\dots 2\cdot\frac{B_{2n+3}}{|\underline{2n+4}}\cdot\frac{B_{2n-1}}{|\underline{2n}}$$
$$+\frac{B_{2n+1}}{|\underline{2n+2}}\cdot\frac{B_{2n+1}}{|\underline{2n+2}}=\frac{B_{4n+3}}{|\underline{4n+4}}(4n+5).$$

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